Math 246C Lecture 7 Notes

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1 Maximal Analytic Continuation and Analytic Functionals

1.1 Maximal analytic continuation

Let X, Y be Riemann surfaces, and let $p : Y \to X$ be holomorphic with no ramification points. Then p is a local biolomorphism, and the pullback map $p^* : O_{X,p(y)} \to O_y$ sending $f \mapsto f \circ p$ is an isomorphism with inverse p_*. Let $\varphi \in O_{X,a}$ for some $a \in X$.

Definition 1.1. An **analytic continuation** of φ is given by (Y, p, f, b), where $p : Y \to X$ is holomorphic and unramified, $f \in \mathrm{Hol}(Y)$, $b \in p^{-1}(a)$, and $p_*(f_b) = \varphi$.

Definition 1.2. An analytic continuation is **maximal** if the following property holds: if (Z, q, g, c) is another continuation of φ, then there exists a holomorphic map $F : Z \to Y$ which is fiber preserving ($p \circ F = q$) such that $F(c) = b$ and $F^* f = g$.

Theorem 1.1. *Let X be a Riemann surface, $\varphi \in O_{X,a}$. Then there exists a maximal analytic continuation (Y, p, f, b) of φ.*

Remark 1.1. One can show that this is unique up to holomorphic diffeomorphism, but we will not do that here.

Lemma 1.1. *Let (Y, p, f, b) be an analytic continuation of φ. Let $\gamma : [0, 1] \to Y$ be a path in Y from b to $y \in Y$. Then the germ $\psi = p_*(f_y) \in O_{X,p(y)}$ is an analytic continuation of φ along the path $p \circ \gamma$.*

Proof. Set $\varphi_t = p_*(f_{\gamma(t)}) \in O_{x,p(\gamma(t))}$ for all $0 \leq t \leq 1$. Then $\varphi_0 = \varphi$, and $\varphi_q = \psi$. We need to check that $[0, 1] \to O_X$ sending $t \mapsto \varphi_t$ is continuous. Let $t_0 \in [0, 1]$. Then there exist neighborhoods $V \subseteq Y$ of $\gamma(t_0)$ and $U \subseteq X$ of $p(\gamma(t_0))$ such that $p|_V : V|toU$ is a holomorphic bijection. Let $g = f \circ ((p|_V)^{-1}) \in \mathrm{Hol}(U)$. Then $p_*(f_z) = g_{p(z)}$ for all $z \in V$. We can find a neighborhood I_{t_0} of t_0 such that $\gamma(I_{t_0}) \subseteq V$. Then for every $t \in I_{t_0}$, $\varphi_t = g_{p(\gamma(t))}$. Thus, ψ is an analytic continuation of φ along $p \circ \gamma$. $\square$

Now let's prove the theorem.

Proof. Let Y be the connected component in O_X containing φ. Then $Y \subseteq O_X$ is open (since O_X is locally connected), and the map $p = p|_Y$ is a local homeomorphism $Y \to X$. There exists a unique complex structure on Y such that $p : Y \to X$ is holomorphic. Let $\zeta \in Y$. Then ζ is a germ of a holomorphic function on X at $p(\zeta)$. Define $f(\zeta) = \zeta(p(\zeta))$. Then $f \in \mathrm{Hol}(Y)$, and if $b = \varphi$, then $b \in p^{-1}(a)$ and $p_*(f_b) = \varphi$.

Let us check the maximality of (Y, p, f, b). Let (Z, q, g, c) be an analytic continuation of φ. Let $z \in Z$ and $z = q(z)$. The germ $q_*(g_z) \in O_{X,x}$ arises by analytic continuation of φ along a curve from a to x in X. Thus, there exists a unique $\psi \in Y$ such that $q_*(g_z) = \psi$. We get a map $F : Z \to Y$ sending $z \mapsto \psi$, and it follows that (Y, p, f, b) is maximal. $\square$

1.2 Analytic functionals and the Fourier-Laplace transform

Definition 1.3. We say that a linear map $\mu : \mathrm{Hol}(\mathbb{C}) \to \mathbb{C}$ is an **analytic functional** if it is continuous in the following sense: there exist a compact $K \subseteq \mathbb{C}$ and constant $C > 0$ such that $|\mu(f)| \leq C \sup_K |f|$ for all $f \in \mathrm{Hol}(\mathbb{C})$.

Remark 1.2. By the Hahn-Banach theorem, μ can be extended to a linear continuous functional on $C(K)$. Then there exists a measure ν on K such that $\mu(f) = \int_K f(z)\, \nu(z)$ for $f \in \mathrm{Hol}(\mathbb{C})$.

Example 1.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a C^1 path, and define the functional $\mu(f) = \int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t))\gamma'(t)\, dt$. μ does not change if γ if replaced by a homotopic path. So the representing measure need not be unique.

Example 1.2. Let $\mu(f) = f^{(j)}(0)$ for $j \geq 0$ is an analytic functional.

Definition 1.4. A compact set $K \subseteq \mathbb{C}$ is called a **carrier** for the analytic functional μ if for every open neighborhood ω of K, there is a constant C_ω such that $|\mu(f)| \leq C_\omega \sup_\omega |f|$ for $f \in \mathrm{Hol}(\mathbb{C})$.

Remark 1.3. The first example shows that carriers need not be unique, either.

Definition 1.5. The **Fourier-Lapclace transform** $\widehat{\mu}$ of μ is defined by

$$\widehat{\mu}(\zeta) = \mu_z(e^{z\zeta}), \qquad \zeta \in \mathbb{C}.$$

We have that $\widehat{\mu}$ is entire (by its description as integration of this function against a measure).

Proposition 1.1. *The map $\mu \mapsto \widehat{\mu}$ is injective.*

Proof. If $\widehat{\mu}(\zeta) = 0$ for all ζ, then $0 = \partial_\zeta^j \widehat{\mu}|_{\zeta=0} = \mu(z^j)$ for all j. In particular, for any polynomial p, $\mu(p) = 0$. Polynomials are dense in $\mathrm{Hol}(\mathbb{C})$, so $\mu(f) = 0$ for all $f \in \mathrm{Hol}(\mathbb{C})$. That is, $\mu(f) = 0$. $\square$